A MINIMUM-JERK TRAJECTORY

Neville Hogan (1984a) noted that smoothness can be quantified as a function of jerk, which is the time derivative of acceleration. Hence, jerk is the third time derivative of location (*i.e.*, position). If the location of a system is specified by variable $x(t)$, then the jerk of that system is:

$$\text{jerk } \dddot{x}(t) = \frac{d^3 x(t)}{dt^3}$$

For your CNS to move your hand or some other end effector smoothly from one point to another, it should minimize the sum of the squared jerk along its trajectory. For a particularly trajectory $x_1(t)$ that starts at time t_i and ends at time t_f, you can measure smoothness by calculating a jerk cost:

$$\int_{t=t_i}^{t_f} \dddot{x}_1(t)^2 \, dt$$

Note that jerk cost is a scalar; the expression above assigns a number to the function $x_1(t)$. Hogan wondered what function $x(t)$ most smoothly connects a starting point to a target in a given amount of time. This function $x(t)$, among all possible functions, has the minimum jerk cost. Some people find this fact interesting, others simply find it tedious. If you find yourself in the latter category, you might consider skipping over the remainder of this chapter.

To make the issue concrete, imagine that you wish to move something 10 cm in a 0.5 s period. The object will be at rest at start time and at the end of the movement. You can write this as:

$$\text{Properties of } x(t): \begin{cases} x(0) = 0 & x(0.5) = 10 \\ \dot{x}(0) = 0 & \dot{x}(0.5) = 0 \\ \ddot{x}(0) = 0 & \ddot{x}(0.5) = 0 \end{cases}$$

What trajectory $x(t)$ has the smoothest path? To find $x(t)$, you need to assign a jerk cost to each possible trajectory, and then find the trajectory with the least cost. Mathematically, this calculation corresponds to minimizing the **functional**:

$$H(x(t)) = \frac{1}{2} \int_{t=0}^{0.5} \dddot{x}^2 \, dt$$

(The ½ factor in the front makes the calculations come out a little prettier; otherwise it has no special significance). To find the minimum of this functional, Hogan used a technique called the calculus of variations. The idea resembles finding the minimum of a function: you find the derivative of the function with respect to a small perturbation and when that derivative is zero, you have found a minimum. The variation is a function that you can name $\eta(t)$.

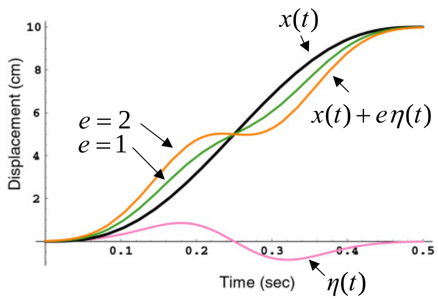


Figure 1 shows an example of $\eta(t)$. The variation has the special property that it smoothly goes away at the boundary conditions, *i.e.*, at the beginning and end of movement:

$$\text{Properties of } \eta(t): \begin{cases} \eta(0)=0 & \eta(0.5)=0 \\ \dot{\eta}(0)=0 & \dot{\eta}(0.5)=0 \\ \ddot{\eta}(0)=0 & \ddot{\eta}(0.5)=0 \end{cases}$$

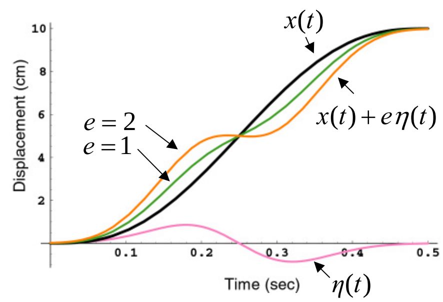


Figure 1. ***A function*** $x(t)$ ***and variation*** $\eta(t)$.

To minimize $H(x(t))$, you can replace $x(t)$ by a variation $x(t) \mapsto x(t) + e\eta(t)$ and you can find the derivative of the new functional with respect to the variation.

$$H(x(t)) = \frac{1}{2}\int_0^{0.5} \dddot{x}(t)^2 dt$$

$$x(t) \mapsto x(t) + e\eta(t)$$

$$H(x+e\eta) = \frac{1}{2}\int_0^{0.5} (\dddot{x} + e\dddot{\eta})^2 dt$$

$$\frac{dH(x+e\eta)}{e} = \int_0^{0.5} (\dddot{x} + e\dddot{\eta})\dddot{\eta}\ dt$$

$$\left.\frac{dH(x+e\eta)}{e}\right|_{e=0} = \int_0^{0.5} \dddot{x}\dddot{\eta}\ dt$$

Using integration by parts, you can rewrite this integral as:

$$\int_0^{0.5} \dddot{x}\dddot{\eta}\ dt = \int_0^{0.5} u\ dv = uv\Big|_0^{0.5} - \int_0^{0.5} v\ du$$

$$u = \dddot{x},\ \ dv = \dddot{\eta}\ dt,\ \ du = x^{(4)}dt,\ \ v = \ddot{\eta}$$

$$\int_0^{0.5} \dddot{x}\dddot{\eta}\ dt = \dddot{x}\ddot{\eta}\Big|_0^{0.5} - \int_0^{0.5} \ddot{\eta}\, x^{(4)}dt = -\int_0^{0.5} \ddot{\eta}\, x^{(4)}dt$$

where $x^{(4)}$ means 4th derivative of function $x(t)$. Continuing the integration by parts,

$$-\int_0^{0.5} \ddot{\eta}\, x^{(4)} dt = -\int_0^{0.5} u\ dv = -uv\Big|_0^{0.5} + \int_0^{0.5} v\ du$$

$$u = x^{(4)},\quad dv = \ddot{\eta}\ dt,\quad du = x^{(5)} dt,\quad v = \dot{\eta}$$

$$-\int_0^{0.5} \ddot{\eta}\, x^{(4)} dt = -x^{(4)}\dot{\eta}\Big|_0^{0.5} + \int_0^{0.5} \dot{\eta}\, x^{(5)} dt = \int_0^{0.5} \dot{\eta}\, x^{(5)} dt$$

$$\int_0^{0.5} \dot{\eta}\, x^{(5)} dt = x^{(5)}\eta\Big|_0^{0.5} - \int_0^{0.5} \eta\, x^{(6)} dt = -\int_0^{0.5} \eta\, x^{(6)} dt$$

This final integral is the derivative of your functional, and you have:

$$\left.\frac{dH(x+e\eta)}{e}\right|_{e=0} = -\int_0^{0.5} \eta x^{(6)} dt \equiv 0$$

The above property must hold true for any function $\eta(t)$, and therefore you have the fact that

$$x^{(6)} = 0\,,$$

which means that some function $x(t)$ that happens to have its sixth derivative equal to zero will minimize your jerk function. The differential equation $x^{(6)} = 0$ has the general solution of:

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

with derivatives:

$$\dot{x}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$

$$\ddot{x}(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3$$

To find the constants, you can plug in what you know about $x(t)$ at the boundaries:

$$x(0) = 0 \to a_0 = 0,\quad x(0.5) = 10 \to a_3(0.5)^3 + a_4(0.5)^4 + a_5(0.5)^5 = 10$$

$$\dot{x}(0) = 0 \to a_1 = 0,\quad \dot{x}(0.5) = 0 \to 3a_3(0.5)^2 + 4a_4(0.5)^3 + 5a_5(0.5)^4 = 0$$

$$\ddot{x}(0) = 0 \to a_2 = 0,\quad \ddot{x}(0.5) = 0 \to 6a_3(0.5) + 12a_4(0.5)^2 + 20a_5(0.5)^3 = 0$$

$$a_3 = 800, a_4 = -2400, a_5 = 1920$$

Thus, you arrive at the function that most smoothly travels 10 cm in 0.5 s:

$$x(t) = 800t^3 - 2400t^4 + 1920t^5 \quad 0 \le t \le 0.5\,\text{s}$$

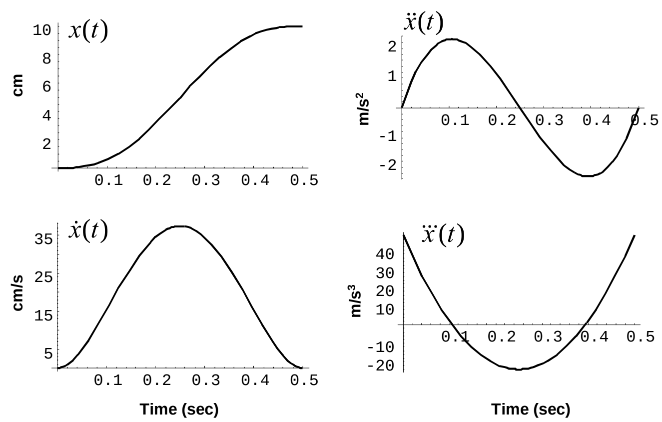


Figure 2 plots this function along with its first three derivatives: velocity, acceleration, and jerk. The function $x(t)$ represents the minimum jerk trajectory in one dimension. Hogan noted that, in general, if something you wanted to move something from location $x = x_i$ to $x = x_f$ in $t = d$ seconds, the minimum jerk trajectory would be:

$$x(t) = x_i + (x_f - x_i)\left(10(t/d)^3 - 15(t/d)^4 + 6(t/d)^5\right) \tag{1}$$

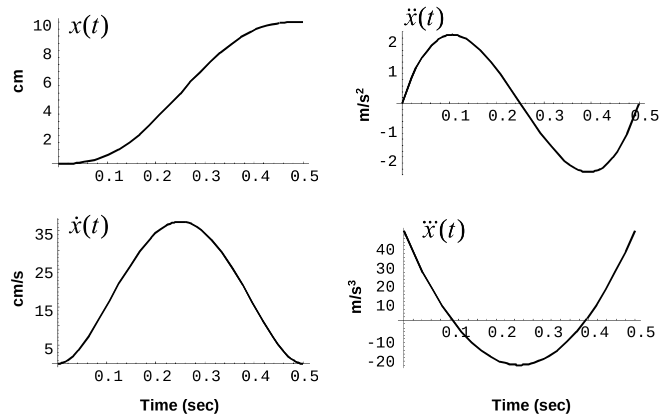


Figure 2. ***Minimum jerk trajectory****. The function* $x(t)$ *that has the minimum jerk cost function as it travels 10 cm over 0.5 seconds.*

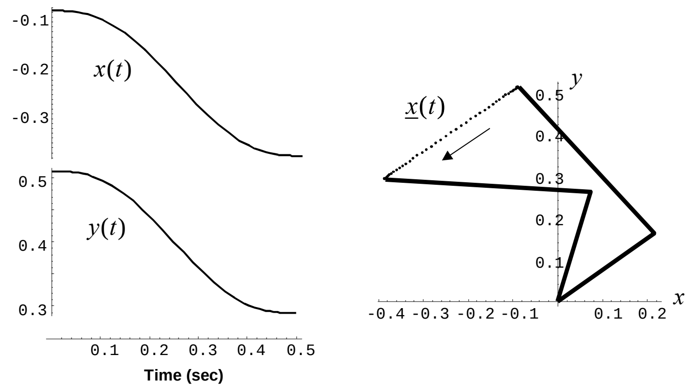
Flash and Hogan (1985) found that, for end-effector locations specified as a vector of two or more dimensions, Eq. (1) described the minimum jerk trajectory for each dimension. For example, for a movement in two dimensions, the functional to minimize is:

$$H(\underline{x}(t)) = \frac{1}{2}\int_{t=0}^{a} (\dddot{x}^2 + \dddot{y}^2)dt$$

and the minimum jerk trajectory in two dimensions is:

$$\underline{x}(t) = \begin{bmatrix} x_i + (x_f - x_i)\left(10(t/a)^3 - 15(t/a)^4 + 6(t/a)^5\right) \\ y_i + (y_f - y_i)\left(10(t/a)^3 - 15(t/a)^4 + 6(t/a)^5\right) \end{bmatrix} \tag{2}$$

Eq. (2) implies that a minimum jerk trajectory in two or three dimensions always corresponds to a



straight line.

Figure 3 exemplifies this relationship, for a two-joint arm moving from an initial- to a final location in 0.5 s. Note how each component of location in cartesian coordinate moves smoothly to its final value and end-effector location moves along a straight line.

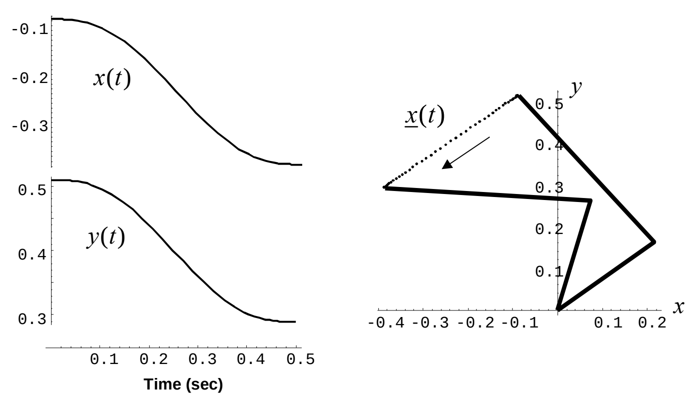


Figure 3. ***A minimum jerk motion in two dimensions is a straight line****. The end-effector location of a two-link arm is moved from (-0.09, 0.51) to (-0.39, 0.29) in 0.5 seconds. The trajectory of x- and y- components of end-effector location are plotted, as well as end-effector location.*

Why not minimum snap, crackle, or pop?

The third derivative of location with respect to time is called jerk. The fourth, fifth, and sixth derivatives are called snap, crackle, and pop, respectively. How can you know that a minimum-jerk description provides the best description of your reaching movements: Why not minimum snap?

To answer this question, Magnus Richardson and Tama Flash (2002) considered how $x(t)$ changed as a function of *n* in the following expression:

$$H(x(t)) = \frac{1}{2}\int_{t_i}^{t_f} \left(\frac{d^n x}{dt^n}\right)^2 dt \tag{3}$$

They found that as the order of the derivative *n* increased, the solution to the functional $x(t)$ approached a step function.

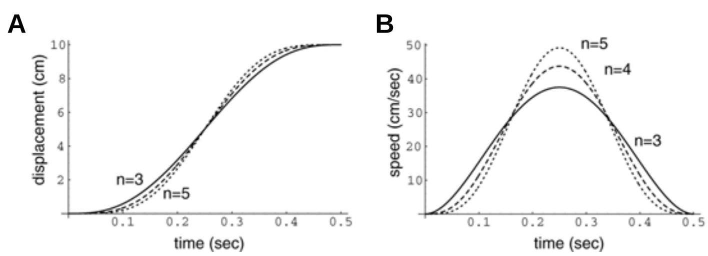


Figure 4A shows the minimum jerk, snap, and crackle trajectories. Note how the first derivative (speed) of each trajectory becomes narrower and taller as you minimize jerk, snap and crackle. Therefore, if you wish to minimize snap, the fourth derivative of location, you get a movement with a higher peak speed than a trajectory that minimizes jerk. This means that as you increase *n* in Eq. (3), the solution yields a trajectory with a larger peak speed relative to average speed.

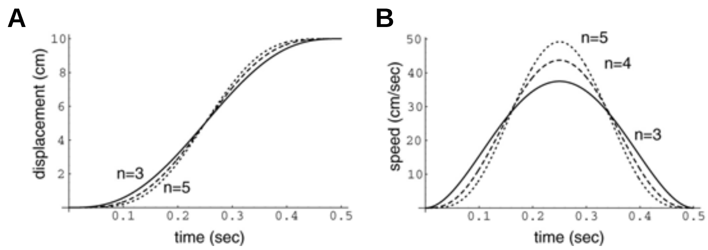


Figure 4. ***Movements with minimum jerk, snap, or crackle****.* ***A****. A minimum jerk (n=3), a minimum snap (n=4), and a minimum crackle (n=5) trajectory. The variable n refers to the order of the derivative in Eq.(3).* ***B****. Speed of the movement for each trajectory. Note that the ratio of peak speed to average speed increases as n increases.*

If you call this ratio of peak speed to average speed r, then a minimum-acceleration trajectory [*i.e.*, where $n = 2$ in Eq. (3)], has a ratio of $r = 1.5$. For a minimum-jerk trajectory, $n = 3$ and $r = 1.875$; for a minimum-snap trajectory, $n = 4$ and $r = 2.186$. Psychophysical experiments reveal that your reaching movements have a ratio r that is about 1.75, and thus most resemble minimum-jerk trajectories (Flash and Hogan 1985).

Smooth trajectories via a feedback controller

The minimum jerk trajectory in Eq. (2) describes how a system should move from rest to a target location in a desired time. It is like a feed forward controller that describes the desired behavior of a system without taking into account feedback during the motion. Using this approach, it is not clear how we should proceed if the target happened to jump halfway during the movement, or the limb was perturbed.

To address these issues, Bruce Hoff and Michael Arbib (1992) reformulated the solution to the functional so that the result was a feedback control system. This system monitored both the location of the hand and the target and ensured that the current desired change in hand location was always such that it brought the hand in a minimum jerk path to the target. Here we summarize their approach.

Recall that the general solution to the minimum jerk functional was of the form $x^{(6)} = 0$, which implies that the trajectory is a fifth order polynomial. Let us normalize our measure of time so that it is 0 when we start the movement and 1 when we reach the target. If τ represents this normalized time, we have:

$$\tau = \frac{t - t_0}{D} \qquad D = t_f - t_0$$

Our minimum jerk trajectory has the form:

$$x(t) = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4 + a_5\tau^5$$

Note that $\frac{d\tau}{dt} = \frac{1}{D}$ and therefore:

$$\dot{x}(t) = \frac{a_1}{D} + \frac{2a_2}{D}\tau + \frac{3a_3}{D}\tau^2 + \frac{4a_4}{D}\tau^3 + \frac{5a_5}{D}\tau^4$$

$$\ddot{x}(t) = \frac{2a_2}{D^2} + \frac{6a_3}{D^2}\tau + \frac{12a_4}{D^2}\tau^2 + \frac{20a_5}{D^2}\tau^3$$

Rather than assuming that the movement begins from rest, we assume that our initial conditions are more generally specified as follows:

$$x(t_0) = x_i \quad \dot{x}(t_0) = v_i \quad \ddot{x}(t_0) = p_i$$

At $t = t_0$, we have $\tau = 0$. Therefore, we have:

$$a_0 = x_i \quad a_1 = Dv_i \quad a_2 = \frac{Dp_i}{2}$$

Our conditions at the end of the movement are:

$$x(t_f) = x_f \quad \dot{x}(t_f) = 0 \quad \ddot{x}(t_f) = 0$$

At $t = t_f$, we have $\tau = 1$. Therefore, we have:

$$a_3 = \frac{-3D^2}{2}p_i - 6Dv_i + 10(x_f - x_i)$$

$$a_4 = \frac{3D^2}{2}p_i + 8Dv_i - 15(x_f - x_i)$$

$$a_5 = -\frac{D^2}{2}p_i - 3Dv_i + 6(x_f - x_i)$$

This gives us an expression for $x(t)$ that is valid for any initial condition. For example, at any time into the movement t, label that time $t = t_0$, and assume that we are at state $q = \begin{bmatrix} x_i & v_i & p_i \end{bmatrix}^T$. The **change** that should occur in our acceleration is specified by $\dddot{x}$:

$$\dddot{x}(t) = \frac{6a_3}{D^3} + \frac{24a_4}{D^3}\tau + \frac{60a_5}{D^3}\tau^2$$

At $t = t_0$, $\tau = 0$, and $D = t_f - t$. Therefore

$\dddot{x}(t_0) = \frac{6a_3}{D^3} = \frac{60}{D^3}(x_f - x(t_0)) - \frac{36}{D^2}\dot{x}(t_0) - \frac{9}{D}\ddot{x}(t_0)$. If we write a control law as:

$\dot{q} = Aq + Bx_f$, then we have:

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-60}{D^3} & -\frac{36}{D^2} & -\frac{9}{D} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{60}{D^3} \end{bmatrix} x_f$$

Given the current position of the hand with respect to target, and the hand’s velocity and acceleration, the above expression provides a method for calculating a desired change in hand position, velocity, and acceleration, so that the hand arrives at the target in an optimally smooth way.

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